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NOTE

Huygens' Solution to the Gambler's Ruin Problem

EDDIE SHOESMITH

University of Buckingham, Buckingham MK18 1EG, England

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At the end of his famous treatise on probability, "De ratiociniis in ludo aleae," Huygens [1657] posed five problems for his readers to solve. The fifth of these, whose origin and early solutions have been described by Edwards [1983], has become known as the "gambler's ruin" problem. In John Arbuthnot's translation of Huygens' tract, the problem was rendered as follows:

> A and B taking 12 counters, each play with three dice after this manner, that if 11 comes up, A shall give one counter to B; but if 14 comes up, B shall give one to A, and that he shall gain who first has all the counters. A's hazard to B's is 244140625 to 282429536481. [Arbuthnot 1692]

The problem for the reader was to provide a justification for the odds ratio given.

The general solution to the gambler's ruin problem is that, if the odds ratio (and hence the ratio of probabilities) at each throw of the set of dice (comprising an arbitrary, fixed number of dice) is c/d, and if each player starts with m counters, then the odds ratio for winning the entire game is c^m/d^m. Putting $m = 12$ and $c/d = 15/27$ gives Huygens' final odds ratio of 244140625/282429536481 (15 and 27 being, respectively, the "number of chances" of scoring 11 and 14 in a single throw of three dice).

The present note examines Huygens' own worked solution, which is contained in Volume 14 of his *Collected Works* [Huygens 1920, 151–155]. The working did not achieve complete generality, but did include an intriguing diagrammatic component. I suggest that this was a device Huygens used to indicate the structure of an essentially inductive algebraic argument.

The note in [Huygens 1920, 151–155] originated on a sheet of paper, dated 1676, and was one of the solutions discussed by Edwards [1983] in a paper concerned largely with the work of Fermat and Pascal on the same problem. One of the features of Edwards' paper is his interpretation of the solutions given by these two in terms of their respective "styles" with regard to probability calculations [Edwards 1982, 1983]. He argued that Fermat's characteristic approach was that of exhaustive enumeration, so that comparison of "numbers of chances" was his

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natural mode of measurement; whereas Pascal, who was more eclectic in his approach, used the concept of "value" of a chance or of a game if it provided a more elegant solution to the problem in hand. In particular, calculations in terms of "values" proved to be more useful than enumeration for gaming problems where the game could, in principle, proceed *ad infinitum*, like the one in Huygens' fifth problem.

In terms of style, Huygens was very much inclined to use algebraic or arithmetical methods that were focused on the "value" of the game to one or the other player: or, as we might now say, the player's expectation (in the original Latin of Huygens' note on the present problem, this concept is usually rendered as *spes*). His worked solution to the gambler's ruin problem was no exception. It was presented, however, in a compact, fragmentary fashion, leaving doubt as to his detailed mode of reasoning.

Let the expectation of player B at a given stage in the game be represented by $E\{t, 0\}$ if the net transfer of counters is t in favor of player B, and $E\{0, t\}$ if the transfer is in player A's favor. At the start of the game, therefore, player B's expectation will be denoted $E\{0, 0\}$, as it is also at any other juncture in the game when the transfer of counters between players has had no net effect. Huygens was seeking to relate $E\{0, 0\}$ to $E\{m, 0\}$ and $E\{0, m\}$. He did not provide a reasoned justification for a completely general relationship, but did sketch an inductive argument enabling the solution for a game involving $2m$ points to be derived from that for m points. Hence algebraic solutions for $m = 2$ and $m = 3$ provided, by induction, solutions for $m = 2 \times 2^k$ and $m = 3 \times 2^k$, respectively, with k a natural number in each case. The detailed steps in the inductive argument were not in all cases explicitly set out by Huygens, but are implied by his diagrams if these are interpreted, as seems appropriate, to be statements of relationships between expectations at various stages of the games.

Huygens' diagrams resemble what we now call event-, possibility-, or probability-trees. Edwards [1983, 77] notes that they may be the first on record. I shall refer to them as expectation-trees for the reason that I have outlined above. Figure 1a shows the form (though not the notation) of Huygens' diagram for $m = 2$, the case he examined first. The branch segments are labeled with the respective numbers of chances and the node identifiers $\{t, 0\}$ and $\{0, t\}$ indicate the net transfer of counters from one player to the other: $\{2, 0\}$ and $\{0, 2\}$ are therefore winning positions for player B and player A, respectively. Huygens' notation is shown in Fig. 1b, a reproduction from [Huygens 1920, 152]. The figures 9 and 5 refer to the specific case when the problem is posed as in "De ratiociniis in ludo aleae" (for which $d/c = 27/15 = 9/5$; see above). Huygens demonstrated that

$$E\{0, 0\} = (d^2E\{2, 0\} + c^2E\{0, 2\})/(d^2 + c^2). \tag{1}$$

The algebraic working accompanying the diagram indicates that Huygens found expressions for $E\{0, 0\}$ in terms of $E\{1, 0\}$ and $E\{0, 1\}$ (x, y, z, respectively, in his notation), for $E\{1, 0\}$ in terms of $E\{2, 0\}$ and $E\{0, 0\}$ (y, n, x), and for $E\{0, 1\}$ in terms of $E\{0, 0\}$ and $E\{0, 2\}$ (z, x, 0), then eliminated $E\{1, 0\}$ and $E\{0, 1\}$ by direct substitution. For player B the odds ratio for winning the game (as against losing it)

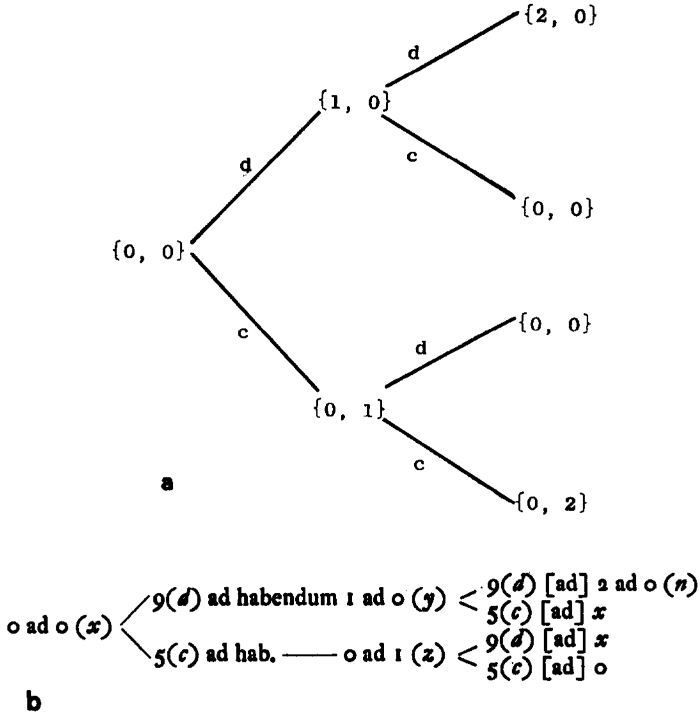

FIG. 1. In this diagram, and in Figs. 2, 4, 5, and 6, part b is reproduced from [Huygens 1920, 151–155].

is d^2/c^2. It is of relevance to note that Huygens labeled his diagram (Fig. 1b) with symbols denoting player B's expectation, as well as the game score, at each node. Further, he assisted the argument by producing the diagram in Fig. 2 (my notation in part a; reproduction of Huygens' in part b), in which the terminal nodes were labeled with expectations, but not game scores. This was clearly intended to show the expectational relationships implied by the game structure.

I suggest that in Huygens' working, the result (1) was not solely a solution for $m = 2$, but an algebraic identity that could be extended by analogy to any portion of an expectation-tree with the structure shown in Fig. 3. In this diagram I have, for simplicity of expression, adopted a notation similar to that of Edwards [1983, 75], letting E_u denote player B's expectation when player A needs u straight points to win the game. $E\{t, 0\}$ is therefore equivalent to E_u with $u = m + t$, and $E\{0, t\}$ to E_u with $u = m - t$. We have, from Fig. 3,

$$E_u = (sE_{u+i} + rE_{u-i})/(s + r)$$

$$E_{u+i} = (sE_{u+2i} + rE_u)/(s + r)$$

$$E_{u-i} = (sE_u + rE_{u-2i})/(s + r);$$

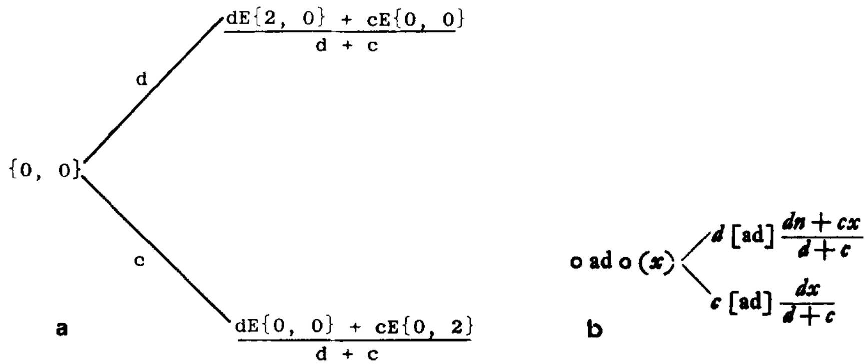

FIGURE 2

so, elimination of E_{u+i} and E_{u-i} from the first of these equations by substitution of the second and third gives

$$E_u = (s^2E_{u+2i} + r^2E_{u-2i})/(s^2 + r^2). \tag{2}$$

Although Huygens did not write down an identity in a form as general as this, its use is implicit in the perfunctory argument that followed for $m = 4$, $m = 8$, and so on.

Huygens represented the 4-points game by the tree shown in Fig. 4 (again, my notation in part a; Huygens' reproduced in part b). This is identical in structure to Fig. 3, with $r = c^2$, $s = d^2$, $u = 4$, $i = 2$; so, it is apparent that (2) gives

$$E\{0, 0\} = (d^4E\{4, 0\} + c^4E\{0, 4\})/(d^4 + c^4).$$

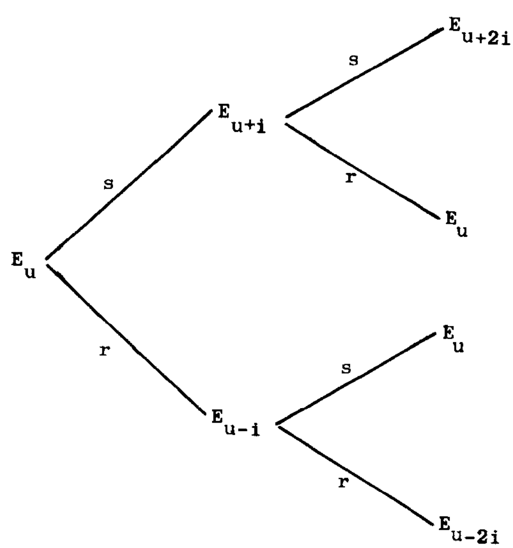

FIGURE 3

The odds ratio for winning as against losing the game, for player B, is d^4/c^4. Huygens gave this result after noting the similarity between Figs. 4b and 1b, with c^2 and d^2 replacing c and d, respectively, but he recorded no further working.

What justification did Huygens conceive for the immediate simplification of the tree for $m = 4$? The full *event*-tree for four tosses of the dice has 16 outcomes, of which Huygens apparently omitted 12 from consideration. Edwards [1983, 77] refers to Huygens' solution for $m = 4$ as "a sort of convolution of the two-points problem into itself," commenting that "the argument is by no means obvious, and since Huygens offers no explanation beyond a diagram we cannot tell exactly what form it took in his mind." The likely mode of reasoning becomes clearer, however, if Fig. 4 is interpreted to be an *expectation*-tree, a summary of the expectational identities required to provide a solution for $m = 4$. These identities are derived by applying (2) (with $r = c$, $s = d$, $i = 1$), or an equivalent relationship, successively, to throws 1 and 2 starting at $\{0, 0\}$ ($u = 4$), to throws 3 and 4 starting at $\{2, 0\}$ ($u = 6$), and to throws 3 and 4 starting at $\{0, 2\}$ ($u = 2$):

$$E\{0, 0\} = (d^2E\{2, 0\} + c^2E\{0, 2\})/(d^2 + c^2)$$

$$E\{2, 0\} = (d^2E\{4, 0\} + c^2E\{0, 0\})/(d^2 + c^2)$$

$$E\{0, 2\} = (d^2E\{0, 0\} + c^2E\{0, 4\})/(d^2 + c^2)$$

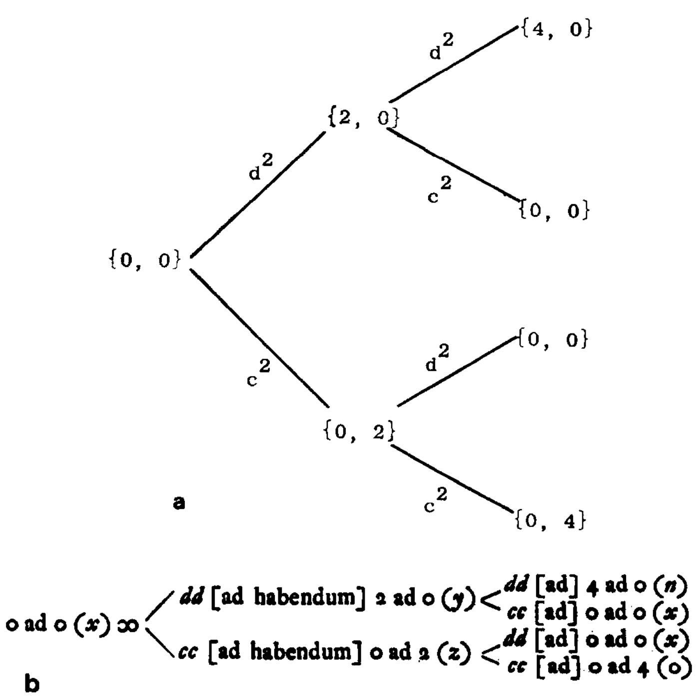

FIG. 4. The modern equivalent of the symbol to the right of "o ad o (x)," which also appears in Figs. 5b and 6b, is "=".

Huygens' expectation-tree (Fig. 4b) was in effect a representation of these three identities, from which $E\{0, 0\}$ was obtained in terms of $E\{4, 0\}$ and $E\{0, 4\}$ by a further round of substitution to eliminate $E\{2, 0\}$ and $E\{0, 2\}$ from the first equation. Equivalently, this can be expressed as yet a further application of (2) (with $r = c^2$, $s = d^2$, $u = 4$, $i = 2$).

As well as giving a solution for $m = 4$, this is confirmation that (2), with $r = c^2$, $s = d^2$, $i = 2$, or an equivalent relationship, can be applied to any portion of an expectation-tree with structure similar to that of the 4-points game. In particular, for a game with $m = 8$, the first four throws starting at $\{0, 0\}$ ($u = 8$), the last four throws starting at $\{4, 0\}$ ($u = 12$), and the last four throws starting at $\{0, 4\}$ ($u = 4$) each have this structure, so application of (2) with $r = c^2$, $s = d^2$, $i = 2$ gives

$$E\{0, 0\} = (d^4E\{4, 0\} + c^4E\{0, 4\})/(d^4 + c^4)$$

$$E\{4, 0\} = (d^4E\{8, 0\} + c^4E\{0, 0\})/(d^4 + c^4)$$

$$E\{0, 4\} = (d^4E\{0, 0\} + c^4E\{0, 8\})/(d^4 + c^4).$$

Then, by elimination of $E\{4, 0\}$ and $E\{0, 4\}$ from the first of these equations,

$$E\{0, 0\} = (d^8E\{8, 0\} + c^8E\{0, 8\})/(d^8 + c^8).$$

Huygens stated this result—in his notation, $x = d^8n/(d^8 + c^8)$—immediately after the result for $m = 4$, without further working or diagrams, and indicated that it could be extended by successive doublings of the "size" of the game. His argument, it seems, was essentially inductive. If the solution

$$E\{0, 0\} = (d^mE\{m, 0\} + c^mE\{0, m\})/(d^m + c^m) \tag{3}$$

holds good for the m-point game, then

$$E\{0, 0\} = (d^{2m}E\{2m, 0\} + c^{2m}E\{0, 2m\})/(d^{2m} + c^{2m}) \tag{4}$$

holds for the $2m$-point game. The structural similarity between the entire m-point game, the first m throws of the $2m$-point game, and the last m throws of the $2m$-point game, starting at either $\{m, 0\}$ or $\{0, m\}$, enables (3) to be extended by analogy to give

$$E\{0, 0\} = (d^mE\{m, 0\} + c^mE\{0, m\})/(d^m + c^m)$$

$$E\{m, 0\} = (d^mE\{2m, 0\} + c^mE\{0, 0\})/(d^m + c^m)$$

$$E\{0, m\} = (d^mE\{0, 0\} + c^mE\{0, 2m\})/(d^m + c^m).$$

This set of three identities then yields (4) by elimination of $E\{m, 0\}$ and $E\{0, m\}$ from the first of the equations. Result (3) evidently is good for $m = 2$, and consequently also for $m = 4, 8$, etc.

To generate a series of solutions for $m = 3, 6, 12$, etc., Huygens found that a little more preliminary algebra was necessary for the case $m = 3$. He began by drawing the tree shown in Fig. 5b (part a of Fig. 5 gives the same tree with the notation I have adopted). Although he did not record the accompanying algebra to

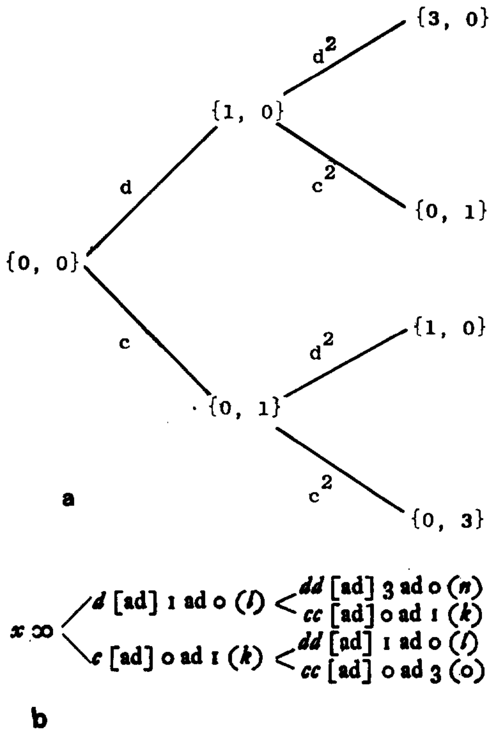

FIGURE 5

justify the simplification of the game structure for the second and third throws, it is apparent that it arises from the same sort of argument and algebraic manipulation that he used consistently throughout the working for $m = 2, 4, 8$, etc. Application of (2), for example, with $r = c$, $s = d$, $i = 1$, to the second and third throws, starting at $\{1, 0\}$ $(u = 4)$ and $\{0, 1\}$ $(u = 2)$, yields

$$E\{1, 0\} = (s^2E\{3, 0\} + r^2E\{0, 1\})/(s^2 + r^2)$$

$$E\{0, 1\} = (s^2E\{1, 0\} + r^2E\{0, 3\})/(s^2 + r^2),$$

which are the expectational relationships depicted on the right in Fig. 5b. Further algebra to eliminate $E\{1, 0\}$ and $E\{0, 1\}$, which Huygens did record in his working note ($E\{1, 0\}$ and $E\{0, 1\}$ were denoted l and k, respectively; see Fig. 5), then gave

$$E\{0, 0\} = (d^3E\{3, 0\} + c^3E\{0, 3\})/(d^3 + c^3).$$

As with the 4-points problem, Huygens illustrated the algebraic working with a tree diagram clearly intended to represent the expectational relationships between $E\{0, 0\}$, $E\{1, 0\}$ and $E\{0, 1\}$. This is reproduced here as Fig. 6b, with Fig. 6a showing the same diagram but with my adopted notation.

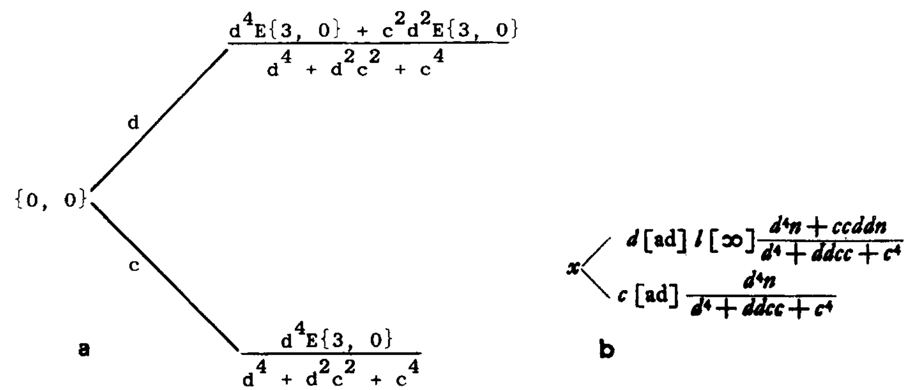

FIGURE 6

Huygens indicated that the result for the 3-points game could be used to provide a solution for the 6-points game, namely that the expectations of players B and A are in the ratio d^6/c^6. Again, it seems likely that he was applying the structural similarity argument to the first three, and last three, throws of the 6-points game.

His working note concluded with the brief comments that the corresponding solution for the 5-points game can be shown to be d^5/c^5, though rather more effort is needed than for the 3-points game, and that in general the solution is given by the ratio (d/c) raised to the power of the number of straight points needed to win the game from its start.

ACKNOWLEDGMENTS

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REFERENCES

Arbuthnot, J. 1962. *Of the laws of chance, or, a method of calculating of the hazards of game*. London.

Edwards, A. W. F. 1982. Pascal and the problem of points. *International Statistical Review* **50**(3), 259–266.

——— 1983. Pascal's problem: The "gambler's ruin." *International Statistical Review* **51**(1), 73–79.

Huygens, C. 1657. De ratiociniis in ludo aleae. In *Exercitationum Mathematicarum Libri quinque*, I. van Schooten, ed. Amsterdam.

——— 1920. *Oeuvres complètes de Christiaan Huygens*, Vol. 14. La Haye: Nijhoff.